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ON THE DYNAMICS EQUATIONS OF SYSTEMS OF INTERCONNECTED BODIES*

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A method is elucidated for deriving the equations of motions of mechanical systems comprised of an arbitrary number of absolutely solid bodies containing both closed and open links. The method proposed is universal in nature and convenient for algorithmization and programming on electronic computers.

The method proposed in /1/ is most general for the description of systems of many bodies with the structure of interconnections with closed links. Its essential disadvantage is, however, the lack of a formal apparatus to describe the couplings in the system and the incomplete utilization of information about the relative motion in the adjacent bodies due to discarding part of the hinges during transformation of the system with closed links into a system with the structure of a tree.

The method elucidated below to derive the dynamical equations can be applied to mechanical systems that are sets of bodies connected by holonomic and non-holonomic, scleronomic, and rheonomic constraints. It is assumed that the system of interconnections is such that a system that does not contain a closed link can be obtained by a single slash of the pertinent bodies. To simplify the exposition, the method is demonstrated on scleronomic holonomic systems, often encountered in practice. The proposed method is a development of the formalism elucidated in /2/.

1. Description of the system structure.

A holonomic system consisting of $N+1$ absolutely solid bodies arbitrarily connected by hinges and containing closed and open kinematic links is considered. An example of such a system is the "Coat-a-Matic" manipulator actuator (Fig.1).

The most general meaning is embodied in the concept of a hinge. It is assumed that not more than one hinge exists between two bodies, and each hinge connects just two bodies of the system, which we later designate as neighbors. The motion of one body of the system is taken as given, at first. The number 0 is ascribed to this body. The remaining bodies in the system are numbered from 1 to N arbitrarily. The hinges are also numbered arbitrarily from 1 to n. The presence of closed loops is assumed in the system, consequently, $n > N$.

We represent the system structure by a graph, whose vertices s_i $(i = 0, 1, \ldots, N)$ symbolize the bodies of the system, and the edges u_a $(a = 1, 2, \ldots, n)$ are the hinges. The primary graph of the manipulator actuator (Fig.1) is represented in Fig.2 where one of the possible methods of numbering the system bodies and hinges is also shown.

By opening the closed loops, we transform the primary graph into a graph with the structure of a tree. We assume such an opening can be made by a single bifurcation of appropriate vertices of the primary graph. Upon bifurcation, two images of one vertex remain that are not connected directly, and belong to different branches of the open graph obtained. The new graph, which we shall designate secondary, evidently has as many new vertices as bifurcations. We ascribe numbers from $N+1$ to n to the new vertices. The transformation of the graph in Fig.2 into an open graph can be accomplished by bifurcation of the vertices with numbers 1 and 3, say, (Fig.3a), or with numbers 2 and 3 (Fig.3b). The so-called path between two vertices is determined uniquely in the secondary graph. This is such a sequence of vertices and edges connecting the vertices under consideration that no hinge is negotiated twice.

We orientate the primary graph representing the system structure by selecting a definite direction, denoted by an arrow, on each edge. In selecting the direction on the edge, we start from which of the two adjacent bodies connected by this hinge will be taken as basis with respect to which we will consider the relative motion of the neighboring body. The direction to the edge is given from the vertex mapping the basic body to the vertex mapping the neighboring body for the hinge under consideration. One of the possible orientations of the edges of the graph is represented in Fig.2. We later call the oriented edges arcs.

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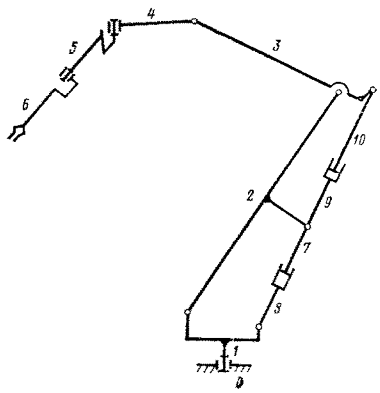

Fig.1

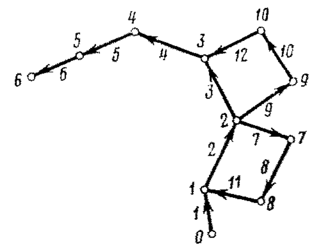

Fig.2

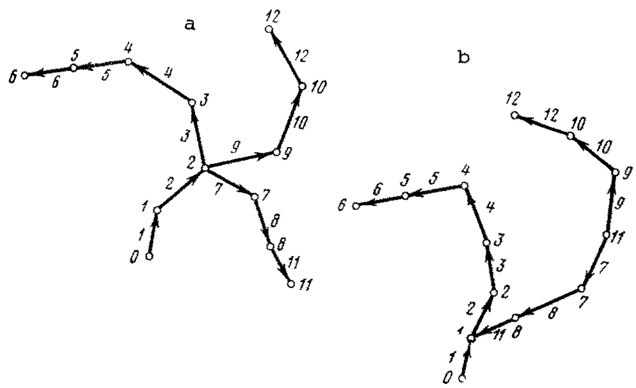

Fig.3

The orientation of the primary graph is carried over without variation to the open secondary graph. The orientation of the secondary graph governs the following two functions: $i^+(a)$ and $i^-(a)$ of argument a $(a = 1, 2, \ldots, n)$ denoting the arc number. The function $i^+(a)(i^-(a))$ determines the number of the vertex from which the arc u_a starts (terminates). For the graphs in Fig.3 the values of these functions equal, respectively

a)	a	1	2	3	4	5	6	7	8	9	10	11	12
	$i^+(a)$	0	1	2	3	4	5	2	7	2	9	8	10
	$i^-(a)$	1	2	3	4	5	6	7	8	9	10	11	12
b)	$i^+(a)$	0	1	2	3	4	5	11	7	11	9	8	10
	$i^-(a)$	1	2	3	4	5	6	7	8	9	10	1	12

By using the functions $i^+(a)$ and $i^-(a)$ we define the following quantities:

$$S_{ia} = \begin{cases} +1, & \text{if } i = i^+(a) \quad (i = 0, 1, \ldots, n), \\ -1, & \text{if } i = i^-(a) \quad (a = 1, 2, \ldots, n), \\ 0 & \text{otherwise} \end{cases}$$

$$T_{ai} = \begin{cases} +1, \text{ if } u_a \text{ belongs to the path between } s_0 \text{ and } s_i \text{ and is directed to } s_0, \\ -1, \text{ if } u_a \text{ belongs to the path between } s_0 \text{ and } s_i \text{ and is directed from } s_0, \\ 0 \text{ otherwise} \end{cases}$$

$(a,\ i = 1, 2, \ldots, n)$.

These quantities determine the matrices

$$S=(S_{ia})_{i=1,\ a=1}^{n,\ \ n},\quad S_0=(S_{0a})_{a=1}^{n}$$
$$T=(T_{ai})_{a=1,\ i=1}^{n,\ \ n}$$

Let S^+ be the matrix obtained from S if all its elements, -1, are replaced by 0. We also introduce the matrix W of dimension $N\times n$ with the elements W_{km}:

$$W_{km}=\begin{cases}+1, & \text{if } k=m,\\ -1, & \text{if } k \text{ is the number of the bifurcated vertex in the primary graph, and } m \text{ is the number of the new vertex in the secondary graph.}\\ 0 & \text{otherwise}\end{cases}$$

The nonzero elements of the matrix W for the graphs in Fig.3 have the form

$$W_{11}=W_{22}=\ldots=W_{10,10}=1$$

and

$$\text{a) } W_{1,11}=W_{3,12}=-1,\quad \text{b) } W_{2,11}=W_{3,12}=-1$$

We extract the sub-matrix I consisting of rows in W corresponding to the numbers of all bifurcated vertices of the primary graph, from W. For the graphs under consideration in Fig. 3, I has the form

$$\text{a) }\begin{Vmatrix}1&0&0&0&0&0&0&0&0&0&-1&0\\0&0&1&0&0&0&0&0&0&0&0&-1\end{Vmatrix}$$
$$\text{b) }\begin{Vmatrix}0&1&0&0&0&0&0&0&0&0&-1&0\\0&0&1&0&0&0&0&0&0&0&0&-1\end{Vmatrix}$$

We also form a matrix H which is obtained from W if we replace the nonzero elements by 1/2 in its rows corresponding to the number of all the bifurcated vertices of the primary graph.

Finally we denote the matrix obtained from W if each of its elements is taken in absolute value by $|W|$.

2. Kinematics of the system. We will use the mathematical apparatus elucidated in /2/ to describe the kinematics of the system. We introduce a local orthonormal reference point $\mathbf{e}_1^{(i)}$, $\mathbf{e}_2^{(i)}$, $\mathbf{e}^{(i)}_3$ with origin at the center of mass C_i of the body $(i=1,2,\ldots,N)$ (Fig.4) into each body of the system under consideration. The point C_0 is arbitrarily chosen for the body of number 0. We denote the radius-vector of the point C_i with respect to the inertial origin at the point O by $R_i\,(i=0,1,\ldots,N)$. We introduce the vector $\mathbf{z}_a$ connecting two fixed points at adjacent bodies for the hinge a and directed exactly as is the arc u_a into the primary oriented graph mapping the system, to describe the relative motion in hinge number $a\ (a=1,2,\ldots,n)$. The mentioned points are called hinge points (Fig.4). We later denote the column matrix $(\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)})^T$ by $\mathbf{e}^{(i)}$.

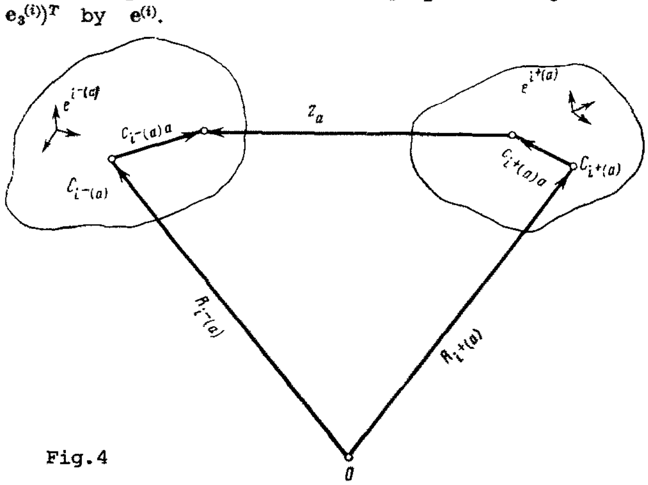

Fig.4

It is convenient to work with a new fictitious system whose structure is represented by the open secondary graph rather than with the original system in considering the kinematics of the system. The former system is obtained if the body to which the bifurcated vertices of the primary graph correspond upon opening of all the closed loops, is imaginatively bifurcated. The body is bifurcated with complete geometric identity conserved between the original and the image. In particular, the centers of mass and basis reference points are identically arranged in both bodies. Joining both bodies with the rest and their numbering are represented in the secondary graph. Here a certain part of the hinged points is conserved in any of the images, namely, the hinge points of those hinges whose corresponding arcs in the secondary graph connect the vertex corresponding to this image to the other vertices. The rest of the hinge points of the original figure in the other image. We consequently obtain a system consisting of $n+1$ bodies and having no closed loops. We call this system an extended system.

The orientation of the reference point $\mathbf{e}^{(i^+(a))}$ relative to the reference point $\mathbf{e}^{(i^-(a))}$ is given by the transition matrix G_a

$$\mathbf{e}^{(i^-(a))} = G_a \mathbf{e}^{(i^+(a))}$$

The quantities $\mathbf{z}_a$ and G_a are functions of the coordinates $q_a = (q_{a1}, q_{a2}, \ldots, q_{an_a})^T$ $(n_a \leqslant 6)$, giving the relative motion in hinge number a. For the bodies of the extended system the coordinate systems are located exactly as in the corresponding bodies of the system. It can be shown that the matrices for the transition of G_{ij} from the coordinate system of the body j to the coordinate system of the body i

$$\mathbf{e}^{(i)} = G_{ij} \mathbf{e}^{(j)}$$

are related to the matrices G_a introduced, by means of the dependence

$$G_{ij} = \prod_{a=1}^{n} G_a^{(T_{aj} - T_{ai})}$$

that follows from the formulas

$$G_{0i} = \prod_{a=1}^{n} G_a^{T_{ai}}, \quad G_{ij} = G_{i0} G_{0j}$$

The relative translation velocity $\mathbf{z}_a^{\circ}$ and the acceleration $\mathbf{z}_a^{\circ\circ}$, the relative angular velocity $\mathbf{\Omega}_a$ and the angular accelerations $\mathbf{\Omega}_a^{\circ}$ $(a = 1, 2, \ldots, n)$ for which the following expressions are true /2/:

$$\mathbf{z}_a^{\circ} = \mathbf{z}_a' q_a^{\cdot}, \quad \mathbf{z}_a^{\circ\circ} = \mathbf{z}_a'^T q_a^{\cdot\cdot} + q_a^{\cdot T} \mathbf{z}_a'' q_a^{\cdot}$$
$$\mathbf{\Omega}_a = \mathbf{p}_a^T q_a^{\cdot}, \quad \mathbf{\Omega}_a^{\circ} = \mathbf{p}_a^T q_a^{\cdot\cdot} + q_a^{\cdot T} \mathbf{p}_a' q_a^{\cdot}$$

are also the characteristics of the relative motion of adjacent bodies, where $\mathbf{z}_a'$, $\mathbf{p}_a$ are vector column matrices of dimension $n_a \times 1$ whose elements are the vectors

$$(\mathbf{z}_a')_i = \frac{\partial \mathbf{z}_a}{\partial q_{ai}}, \quad \mathbf{p}_a = (\mathbf{p}_{a1}, \mathbf{p}_{a2}, \ldots, \mathbf{p}_{an_a})^T$$
$$(i = 1, 2, \ldots, n_a;\ a = 1, \ldots, n)$$

while $\mathbf{z}_a''$, $\mathbf{p}_a'$ are vector matrices of dimension $n_a \times n_a$ with the elements

$$(\mathbf{z}_a'')_{ij} = \frac{\partial^2 \mathbf{z}_a}{\partial q_{ai} \partial q_{aj}}, \quad (\mathbf{p}_a')_{ij} = \frac{\partial \mathbf{p}_{ai}}{\partial q_{aj}}, \quad (i, j = 1, 2, \ldots, n_a;\ a = 1, \ldots, n)$$

We denote the local radius-vectors of the hinge points by $\mathbf{c}_{i^+(a)a}$ and $\mathbf{c}_{i^-(a)a}$ (Fig.4). Introducing the matrices $\mathbf{C}, \mathbf{C}_0$ and $\mathbf{C}^*$ with the elements

$$(\mathbf{C})_{ia} = S_{ia} \mathbf{c}_{ia}, \quad (\mathbf{C}_0)_a = S_{0a} \mathbf{c}_{0a}$$
$$(\mathbf{C}^*)_{ia} = S_{ia}^{+} \mathbf{z}_a \quad (i, a = 1, 2, \ldots, n)$$

we obtain the following expression for the column matrix of the radius-vectors $\mathbf{R}_i$ $(i = 1, 2, \ldots, n)$ of the mass centers of the bodies in the extended system /2/:

$$\mathbf{R} = \mathbf{R}_0 1_n - (\mathbf{C}T)^T 1_n - T^T \mathbf{z} - (\mathbf{C}_0 T)^T$$

Here $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n)^T$ and $1_n = (1, 1, \ldots, 1)^T$ is a column matrix of dimension $n \times 1$ all of whose elements are ones.

Performing the necessary operations on the block matrices, we obtain dependences for all the other absolute kinematic characteristics of the extended system

$$\mathbf{R}^{\cdot\cdot} = T^T [\operatorname{diag} \mathbf{p} T \times (\mathbf{C} + \mathbf{C}^*) - \operatorname{diag} \mathbf{z}']^T q^{\cdot\cdot} + \mathbf{u} \tag{2.1}$$
$$\boldsymbol{\omega} = -T^T \operatorname{diag} \mathbf{p}^T q^{\cdot} + \boldsymbol{\omega}_0 1_n$$
$$\boldsymbol{\omega}^{\cdot} = -T^T (\operatorname{diag} \mathbf{p}^T q^{\cdot\cdot} + \mathbf{f}) + \boldsymbol{\omega}_0^{\cdot} 1_n$$
$$\delta \mathbf{R} = -T^T [(\mathbf{C} + \mathbf{C}^*)^T \times T^T \operatorname{diag} \mathbf{p}^T + \operatorname{diag} \mathbf{z}'^T] \delta q$$
$$\delta \boldsymbol{\pi} = -T^T \operatorname{diag} \mathbf{p}^T \delta q$$

where

$$q=(q_1{}^T, q_2{}^T, \ldots, q_n{}^T)^T, \quad \omega=(\omega_1, \omega_2, \ldots, \omega_n)^T$$
$$\delta\mathbf{R}=(\delta\mathbf{R}_1, \delta\mathbf{R}_2, \ldots, \delta\mathbf{R}_n)^T, \quad \delta\pi=(\delta\pi_1, \delta\pi_2, \ldots, \delta\pi_n)^T$$
$$\mathbf{p}=(\mathbf{p}_1{}^T, \mathbf{p}_2{}^T, \ldots, \mathbf{p}_n{}^T), \quad \mathbf{z}'=(\mathbf{z}_1'^T, \mathbf{z}_2'^T, \ldots, \mathbf{z}_n'^T)^T$$
$$\mathbf{u}=-T^T[\mathbf{C}+\mathbf{C}^*)^T \times (T^T\mathbf{f}-\omega_0^{\cdot}1_n)+\mathbf{w}+\mathbf{g}+\mathbf{h}]+\mathbf{R}_0^{\cdot\cdot}1_n-(\mathbf{C}_0^{\cdot\cdot}T)^T$$

Here ω_i is the vector of the absolute angular velocity of the body numbered i in the extended system, and the vectors $\delta\mathbf{R}_i$ and $\delta\pi_i$ describe the absolute variations of the position and orientation of this body. The quasi-diagonal matrix along whose principal diagonal the elements $A_1, \ldots, A_n$ of the quantity A are located, is denoted by the symbol $\operatorname{diag} A$. These elements can be scalars, vectors, matrices, tensors, etc. For instance, the vector matrices $(\mathbf{p}_{a1}, \mathbf{p}_{a2}, \ldots, \mathbf{p}_{an_a})$ are located on the principal diagonal of the quasi-diagonal matrix $\operatorname{diag}\mathbf{p}^T$, and therefore its dimension is $n \times (n_1+n_2+\ldots+n_n)$. The components of the vector matrices $\mathbf{f}, \mathbf{w}, \mathbf{g}, \mathbf{h}$ are:

$$\mathbf{f}_a=q_a^{\cdot T}\mathbf{p}_a'q_a^{\cdot}+\omega_{i+(a)}\times\mathbf{p}_a{}^Tq_a^{\cdot}, \quad \mathbf{w}_a=q_a{}^{\cdot T}\mathbf{z}_a''q_a^{\cdot}$$
$$g_a=\omega_{i+(a)}\times(\omega_{i+(a)}\times\mathbf{c}_{i+(a)a})-\omega_{i-(a)}\times(\omega_{i-(a)}\times\mathbf{c}_{i-(a)a})$$
$$\mathbf{h}_a=2\omega_{i+(a)}\times\mathbf{z}_a'^Tq_a^{\cdot}+\omega_{i+(a)}\times(\omega_{i+(a)}\times\mathbf{z}_a) \quad (a=1,2,\ldots,n)$$

In order to go from the kinematics of the extended system to the kinematics of the original system, it is necessary to write the equations of the constraints expressing the condition that the open loops are closed. This evidently the condition for the agreement of two images of the bifurcated bodies, which reduces to the requirement of equality of the radius-vectors of their mass centers and equality of the transition matrix between them to the unit matrix. If j and k are the numbers of two images of one bifurcated body, then these conditions are written thus

$$\mathbf{R}_j-\mathbf{R}_k=0, \; G_{jk}=E \tag{2.2}$$

It can be shown that these conditions are expressed for the whole system by the following matrix formulas:

$$IT^T(\mathbf{C}^T1_n+\mathbf{z}+\mathbf{C}_0{}^T)=0 \tag{2.3}$$

$$\prod_{a=1}^{n} G_a^{(IT^T)_{ia}}=E \quad (i=1,2,\ldots,n-N) \tag{2.4}$$

The scalar equations corresponding to this system can be obtained by projecting each vector equation on certain coordinate system. For instance, the coordinate system of body number 0 can be taken, but the coordinate system of a body taken from the loop, whose closedness this equation expresses, is more convenient. In particular, the coordinate system of the bifurcated body can be used.

3. Derivation of the equations of motion. The i-th body of the system alters its position in space during variation of the generalized coordinates, which results in displacement of the mass center by the quantity δR_i and rotation of the body around an axis passing at an infinitesimal angle through C_i. The direction of the axis and the magnitude of the angle of this rotation are determined by the infinitesimal vector $\delta\pi_i$.

Using the D'Alembert principle, we find the following expression for the virtual work performed by the forces acting on the system and by the inertial forces

$$\delta\mathbf{R}^{sT}\cdot(F^*-m\mathbf{R}^{\cdot\cdot s})+\delta\pi^{sT}\cdot(\mathbf{M}^*-\mathbf{L}^{\cdot})=0 \tag{3.1}$$
$$\mathbf{R}^s=(\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N)^T), \quad \delta\pi^s=(\delta\pi_1, \delta\pi_2, \ldots, \delta\pi_N)^T$$
$$\mathbf{L}^{\cdot}=(\mathbf{L}_1^{\cdot}, \mathbf{L}_2^{\cdot}, \ldots, \mathbf{L}_N^{\cdot})^T, \quad \mathbf{F}^*=(\mathbf{F}_1^*, \mathbf{F}_2^*, \ldots, \mathbf{F}_N^*)^T$$
$$\mathbf{M}^*=(\mathbf{M}_1^*, \mathbf{M}_2^*, \ldots, \mathbf{M}_N^*)^T$$

The superscript s denotes that the quantity corresponds to bodies of the initial system. The matrix m has the elements $m_{ij}=\delta_{ij}m_i \; (i,j=1,\ldots,N)$, where δ_{ij} is the Kronecker delta, m_i is the mass of the i-th body of the initial system, $\mathbf{L}_i^{\cdot}=\mathbf{J}_i\cdot\omega_i^{\cdot}+\omega_i\times\mathbf{J}_i\cdot\omega_i$ is the absolute derivative of the moment of momentum of the i-th body relative to its center of mass, $\mathbf{J}_i$ is the central tensor of inertia of this body, $\mathbf{F}^*_i$ is the resultant of all the forces acting on the body of number i, and $\mathbf{M}^*_i$ is the resultant moment of all the forces acting on the i-th body relative to its center of mass $(i=1,2,\ldots,N)$.

Quantities related to the second images of the bifurcated bodies are not in (3.1). In order to use the kinematics encompassing the bifurcated body constructed in the previous section, it is necessary to express $\delta\mathbf{R}^s$ and $\delta\boldsymbol{\pi}^s$ in terms of $\delta\mathbf{R}$ and $\delta\boldsymbol{\pi}$. To do this we use the matrix H introduced above

$$\delta\mathbf{R}^s = H\delta\mathbf{R}, \quad \delta\boldsymbol{\pi}^s = H\delta\boldsymbol{\pi}$$

There remains to find the expressions for the block forces and moments. Let $\mathbf{F}_i$ be the resultant of all the external forces acting on the body with number i, and $\mathbf{M}_i$ their moment relative to the center of mass $C_i\,(i = 1, 2, \ldots, N)$. Letting $\mathbf{X}_a$ denote the internal force in the hinge number a that acts on the body with number $i^+(a)$ and performs virtual work, we find the following expression for the resultant of the internal forces acting on a body with number i and performing virtual work:

$$\sum_{a=1}^{n} (|W|S)_{ia}\mathbf{X}_a \quad (i = 1, 2, \ldots, N)$$

We then obtain for $\mathbf{F}^*$

$$\mathbf{F}^* = \mathbf{F} + |W|S\mathbf{X}$$
$$\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N)^T, \quad \mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^T$$

We also introduce into hinge number a the moment of the internal forces $\mathbf{Y}_a$ that perform virtual work and act on the body number $i^+(a)$, taken relative to the end of the vector $\mathbf{z}_a$. It can be shown that

$$\mathbf{M}^* = \mathbf{M} + |W|S\mathbf{Y} + |W|(\mathbf{C} + \mathbf{C}^*) \times \mathbf{X}$$
$$\mathbf{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)^T, \quad \mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n)^T$$

Now, if the dependences obtained are substituted into (3.1), we obtain an equality of the form

$$\begin{aligned} &\delta q^T (Aq^{\cdot\cdot} - B) = 0 \\ &A = [\operatorname{diag}\mathbf{p}T \times (\mathbf{C} + \mathbf{C}^*) - \operatorname{diag}\mathbf{z}']\,TH^T \cdot mHT^T [\operatorname{diag}\mathbf{p}T \times \\ &\quad (\mathbf{C} + \mathbf{C}^*) - \operatorname{diag}\mathbf{z}']^T + \operatorname{diag}\mathbf{p}TH^T \cdot \operatorname{diag}\mathbf{J} \cdot HT^T \operatorname{diag}\mathbf{p}^T \\ &B = [\operatorname{diag}\mathbf{p}T \times (\mathbf{C} + \mathbf{C}^*) - \operatorname{diag}\mathbf{z}']\,TH^T \cdot (\mathbf{F} - mH\mathbf{u}) - \\ &\quad \operatorname{diag}\mathbf{p}TH^T \cdot [\mathbf{M} - \mathbf{V} + \operatorname{diag}\mathbf{J} \cdot (HT^T\mathbf{f} - \boldsymbol{\omega}_0 \mathbf{1}_N)] - \\ &\quad \operatorname{diag}\mathbf{z}'TH^T |W| S \cdot \mathbf{X} - \operatorname{diag}\mathbf{p}TH^T |W| S \cdot \mathbf{Y} + \\ &\quad \operatorname{diag}\mathbf{p}T \cdot [(\mathbf{C} + \mathbf{C}^*)\,TH^T |W| S - H^T |W| (\mathbf{C} + \mathbf{C}^*)] \times \mathbf{X} \\ &\mathbf{V} = (\boldsymbol{\omega}_1 \times \mathbf{J}_1 \cdot \boldsymbol{\omega}_1,\ \boldsymbol{\omega}_2 \times \mathbf{J}_2 \cdot \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_N \times \mathbf{J}_N \cdot \boldsymbol{\omega}_N)^T \end{aligned} \tag{3.2}$$

When the system under investigation does not contain closed kinematic chains, we obtain the equations of motion directly from (3.2), keeping in mind that the components δq are arbitrary and mutually independent. However, in the presence of kinematic constraints, the quantities δq will already be dependent. We obtain these dependences by variating the equations of constraints (2.2) and (2.5). While the direct variation of the equation of constraints (2.3) is elementary, the direct variation of (2.4) is fraught with technical difficulties. In order to cope with this problem, we note that the equations of constraints (2.2) express the identical agreement of the bifurcated bodies during system motion. Hence, it follows, in particular, that changes in their mutual location in absolute space, which are determined by the change in the radius-vectors $\delta\mathbf{R}$ and the angular position $\delta\boldsymbol{\pi}$ are identically zero

$$I\delta\mathbf{R} = \mathbf{0}, \ I\delta\boldsymbol{\pi} = \mathbf{0} \tag{3.3}$$

After substituting the expressions for $\delta\mathbf{R}$ and $\delta\boldsymbol{\pi}$ from the last two relationships (2.1), we obtain

$$\begin{aligned} &\delta q^T [\operatorname{diag}\mathbf{p}T \times (\mathbf{C} + \mathbf{C}^*) - \operatorname{diag}\mathbf{z}']\,TI^T = \mathbf{0}^T \\ &dq^T \operatorname{diag}\mathbf{p}TI^T = \mathbf{0}^T \end{aligned} \tag{3.4}$$

which are evidently variated versions of (2.3) and (2.4). Projecting each of these vector equations in the coordinate systems mentioned at the end of Sect.2, we obtain a system of $6\,(n - N)$ scalar equations of the form

$$\delta q^T K = 0^T \tag{3.5}$$

where $0^T = (0, \ldots, 0)$ is a matrix consisting of zeroes, of dimension $1 \times 6\,(n - N)$ and K is a matrix of dimension $r \times 6\,(n - N)$, $r = n_1 + n_2 + \ldots + n_n$. Let the rank of the matrix K be $r - d$.

Let D_2^T be a minor of dimension $(r-d) \times (r-d)$, which realizes this rank. Furthermore, let D_1^T be a submatrix of the matrix K consisting of the remaining elements of those columns in which the matrix D_2^T is located. Without limiting the generality, by renumbering the elements of the column q if need be, it can be assumed that the minor D_2^T is in the last $(r-d)$ rows of the matrix K. Then (3.5) can be rewritten as

$$\delta q^{(1)T} D_1^T + \delta q^{(2)T} D_2^T = 0^T \tag{3.6}$$

where $q^{(1)}$ denotes the first d elements of the column q, and $q^{(2)}$ is the last $(r-d)$, $q = (q^{(1)T}, q^{(2)T})^T$. Evidently $q^{(1)}$ represents a complete set of independent, i.e., generalized, coordinate systems. Solving (3.6) for $q^{(2)}$, we find

$$\delta q^{(2)} = -D^{-1}{}_2 D_1 \delta q^{(1)} \tag{3.7}$$

If the trivial dependence $\delta q^{(1)} = E_d \delta q^{(1)}$ were added to (3.7), where E_d is the unit matrix of dimension $(d \times d)$, we obtain

$$\delta q = P \delta q^{(1)}, \quad P = \begin{Vmatrix} E_d \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ -D_2^{-1} D_1 \end{Vmatrix} \tag{3.8}$$

Since the constraints are stationary, then we have

$$I\mathbf{R}^{\cdot} = \mathbf{0}, \quad I\omega = \mathbf{0} \tag{3.9}$$

analogously to the dependence (3.3), and

$$q^{\cdot} = P q^{\cdot(1)} \tag{3.10}$$

analogously to (3.8).

Differentiating (3.9) still again with respect to the time, we obtain

$$IT^T \{[\operatorname{diag} pT \times (C + C^*) - \operatorname{diag} z']^T q^{\cdot\cdot} - (C + C^*)^T \times (T^T f - \omega_0 1_n) - w - g - h - C_0^{\cdot\cdot T}\} = 0 \tag{3.11}$$

$$IT^T (\operatorname{diag} p^T q^{\cdot\cdot} + \mathfrak{f}) = \mathbf{0}$$

Projecting (3.11) on the same axes as when obtaining (3.5), we write them in scalar form

$$q^{\cdot\cdot T} K = K_1$$

where K_1 is a matrix independent of $q^{\cdot\cdot}$.

Let Q_1^T be a matrix consisting of those columns of the matrix K_1 the correspond to the columns forming D_1^T and D_2^T, then

$$q^{\cdot\cdot} = P q^{\cdot\cdot(1)} + Q, \quad Q = \begin{Vmatrix} 0_d \\ \cdot\ \cdot\ \cdot \\ D_2^{-1} Q \end{Vmatrix}, \quad 0_d = (0, 0, \ldots, 0)^T \tag{3.12}$$

where 0_d is a matrix of dimension $d \times 1$ consisting of zeroes. Substituting (3.8) and (3.12) into (3.2) and using the fact that $\delta q^{(1)}$ is a column of independent parameters, we obtain

$$(P^T A P) q^{\cdot\cdot(1)} = P^T (B - AQ) \tag{3.13}$$

Equations (3.13), together with the equations of constraints (3.10), are a system of $(r+d)$ first order equations of motion in q $q^{\cdot(1)}$ which can be written as follows:

$$\frac{dq}{dt} = P q^{\cdot(1)}, \quad \frac{dq^{\cdot(1)}}{dt} = (P^T A P)^{-1} P^T (B - AQ) \tag{3.14}$$

The initial conditions for q satisfy the equations of constraints (2.3) and (2.4). The initial conditions for $q^{\cdot(1)}$ can be chosen arbitrarily.

The number of equations (3.14) is not the smallest possible **($2d$)**, but after integration we obtain at once the complete information about not only the global behavior of the system, but also about the relative motion in any hinge.

Ordinarily, in investigating the dynamics of specific engineering systems not only is information about the changes in the independent coordinates $q^{(1)}$ essential for the design and control, but also information about a whole set of parameters q. That fact must be taken into account, that only the number of degrees of freedom of the system is invariant, while the vector of the generalized coordinates $q^{(1)}$ is not determined uniquely. There is definite arbitrariness in its selection, which is governed by the fact that the rank of the matrix K can be realized by different minors. Moreover, the matrix K itself has a different form for different methods of transforming the system with closed chains into a system with the structure of a tree. The form of the matrix K depends also on the selection of the coordinate systems for the projection of the vector equalities (3.4).

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